

HOLOGRAPHIC FORMULA FOR Q -CURVATURE. II

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ABSTRACT. We extend the holographic formula for the critical Q -curvature in [GJ07] to all Q -curvatures. Moreover, we confirm a conjecture of [J09a].

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1. INTRODUCTION AND FORMULATION OF THE MAIN RESULT

The present work is a sequel to [GJ07], which gave a holographic formula for Branson's critical Q -curvature.

The notion of Q -curvature was introduced by Branson in [B95]. Any Riemannian manifold (M, g) of even dimension n comes with a finite sequence Q_2, Q_4, \dots, Q_n of Q -curvatures. The quantity $Q_{2N} \in C^\infty(M)$ arises by

$$P_{2N}(g)(1) = (-1)^N \left(\frac{n}{2} - N \right) Q_{2N}(g) \quad (1.1)$$

through the constant term of the conformally covariant power P_{2N} of the Laplacian constructed in [GJMS92]. Q_{2N} is a curvature invariant of order $2N$. For general metrics, the sequence P_2, P_4, \dots, P_n of GJMS-operators and the associated sequence of Q -curvatures terminate at the *critical* GJMS-operator P_n and the *critical* Q -curvature Q_n , respectively. A subtle point is that, by $P_n(g)(1) = 0$, the critical Q -curvature Q_n is not defined by (1.1). Instead, it arises through a limiting procedure from

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the subcritical Q -curvatures. The main feature which distinguishes the critical Q -curvature from the subcritical ones is its remarkable linear transformation property

$$e^{n\varphi}Q_n(e^{2\varphi}g) = Q_n(g) + (-1)^{\frac{n}{2}}P_n(g)(\varphi), \quad \varphi \in C^\infty(M)$$

under conformal changes of the metric. Finally, in odd dimensions, the sequences P_2, P_4, \dots and Q_2, Q_4, \dots both are infinite. However, in that case, there is no critical Q -curvature.

The lowest order Q -curvatures are given by the explicit formulas

$$Q_2 = \frac{\text{scal}}{2(n-1)} \quad \text{and} \quad Q_4 = \frac{n}{2}J^2 - 2|\mathbf{P}|^2 - \Delta J, \quad (1.2)$$

where we use the notation

$$J = \frac{\text{scal}}{2(n-1)} \quad \text{and} \quad \mathbf{P} = \frac{1}{n-2}(\text{Ric} - Jg)$$

with Δ denoting the non-positive Laplacian. \mathbf{P} is the Schouten tensor. Among all Q -curvatures, these are by far the mostly studied ones. For more information see [BJ09] and [J09a].

The following theorem extends the main result of [GJ07] to all Q -curvatures.

Theorem 1.1. *Assume that $1 \leq N \leq \frac{n}{2}$ for even n and $N \geq 1$ for odd n . Then*

$$4Nc_NQ_{2N} = 2Nv_{2N} + \sum_{j=1}^{N-1} (2N-2j)\mathcal{T}_{2j}^*\left(\frac{n}{2}-N\right)(v_{2N-2j}) \quad (1.3)$$

with $1/c_N = (-1)^N 2^{2N} N!(N-1)!$.

The identities (1.3) express all Q -curvatures Q_{2N} on M in terms of quantities which are associated to corresponding Poincaré-Einstein metrics g_+ on $M \times (0, \varepsilon)$. These are

- the holographic coefficients v_{2j} and
- the coefficients $\mathcal{T}_{2j}(n/2-N)$ of asymptotic expansions of eigenfunctions of the Laplacian Δ_{g_+} for the eigenvalue $N^2 - n^2/4$.

The relevant concepts will be recalled in Section 2. The role of geometry in one higher dimension in Theorem 1.1 motivates to refer to the formulas (1.3) as holographic. Theorem 1.1 (for even n) was formulated as Conjecture 6.9.1 in [J09a].

The paper is organized as follows. In Section 2, we recall basic concepts and establish some identities which will be referred to as the master relations. These seem to be of independent interest since they hold true in a wider setting. In Section 3, we show that the master relations for Poincaré-Einstein metrics imply Theorem 1.1. For this it suffices to combine the master relations for Poincaré-Einstein metrics with the identities (2.7) and (3.6). This method also gives an alternative proof of (1.3) in the critical case. In Section 4, we give an independent proof of the master relation for the Poincaré metric on the unit ball by using hypergeometric functions.

As a byproduct, it yields closed formulas for the Q -curvature polynomials of round spheres.

2. MASTER RELATIONS

Although we shall work in the same framework as in [GJ07], we are going to use the notation of [J09a]. For the convenience of the reader, we recall the basic definitions. For a given metric h on the manifold M of even dimension n , let

$$g_+ = r^{-2}(dr^2 + h_r) \quad (2.1)$$

with

$$h_r = h + r^2 h_{(2)} + \cdots + r^{n-2} h_{(n-2)} + r^n (h_{(n)} + \log r \bar{h}_{(n)}) + \cdots \quad (2.2)$$

be a metric on $M \times (0, \varepsilon)$ so that the tensor $\text{Ric}(g_+) + n g_+$ satisfies the asymptotic Einstein condition

$$\text{Ric}(g_+) + n g_+ = O(r^{n-2}) \quad (2.3)$$

together with a certain vanishing trace condition. These conditions uniquely determine the coefficients $h_{(2)}, \dots, h_{(n-2)}$. They are given as polynomial formulas in terms of h , its inverse, the curvature tensor of h , and its covariant derivatives. The coefficient $\bar{h}_{(n)}$ and the quantity $\text{tr}(h_{(n)})$ are determined as well (here and in the following, traces are meant with respect to h). Moreover, $\bar{h}_{(n)}$ is trace-free, and the trace-free part of $h_{(n)}$ is undetermined. A metric g_+ with these properties is called a Poincaré-Einstein metric with conformal infinity $[h]$. For odd n , the condition (2.3) can be satisfied to all orders by a formal series $h_r = h + r^2 h_{(2)} + r^4 h_{(4)} + \cdots$ with *even* powers, the coefficients of which are uniquely determined by h . For full details see [FG07].

The volume form of g_+ has the form

$$\text{dvol}(g_+) = r^{-n-1} v(r) dr \text{dvol}(h),$$

where $v(r) = \text{dvol}(h_r) / \text{dvol}(h)$. The coefficients in the Taylor series

$$v(r) = 1 + v_2 r^2 + v_4 r^4 + \cdots$$

are known as *renormalized volume* coefficients ([G00], [G09]) or *holographic* coefficients ([J09a], [BJ09]). We set $v_0 = 1$. For even n , the coefficients $v_{2j} \in C^\infty(M)$, $j = 1, \dots, \frac{n}{2}$ are uniquely determined by h and are given by respective local formulas which involves at most $2j$ derivatives of the metric h . For odd n , all elements in the corresponding infinite sequence v_0, v_2, v_4, \dots are uniquely determined by h .

Now we consider asymptotic expansions of eigenfunctions of the Laplace-Beltrami operator of g_+ , i.e., solutions of the equation

$$-\Delta_{g_+} u = \lambda(n - \lambda)u, \quad \lambda \in \mathbb{C}. \quad (2.4)$$

For given $f \in C^\infty(M)$, we consider formal solutions of (2.4) in form of power series

$$r^\lambda a_0 + \sum_{N \geq 1} r^{\lambda+N} a_N \quad \text{with} \quad a_0 = f. \quad (2.5)$$

It turns out that, for even n , the coefficients $a_N(h; \lambda)$ with odd $N \leq n-1$ vanish, and that the coefficients $a_2(h; \lambda), \dots, a_{n-2}(h; \lambda)$ with even indices are uniquely determined by $a_0 = f$. They are given by respective differential operators $\mathcal{T}_{2N}(h; \lambda)$ acting on f . The operators $\mathcal{T}_{2N}(h; \lambda)$ are natural in the metric h and rational in λ . More precisely, any $\mathcal{T}_{2N}(h; \lambda)$ can be written in the form

$$\mathcal{T}_{2N}(h; \lambda) = \frac{1}{2^{2N} N! (\frac{n}{2} - \lambda - 1) \cdots (\frac{n}{2} - \lambda - N)} P_{2N}(h; \lambda) \quad (2.6)$$

with a polynomial (in λ) family $P_{2N}(h; \lambda) = \Delta_h^N + LOT$; we recall that Δ denotes the non-positive Laplacian. Note that (2.6) shows that the poles of $\mathcal{T}_{2N}(\lambda)$ are contained in the set

$$\left\{ \frac{n}{2} - 1, \dots, \frac{n}{2} - N \right\}.$$

Similarly, for odd n , the infinite sequence $\mathcal{T}_2(h; \lambda), \mathcal{T}_4(h; \lambda), \dots$ is uniquely determined. Now for special parameters λ , the families $P_{2N}(h; \lambda)$ contain the GJMS-operators of h . In fact, for even n and $1 \leq N \leq \frac{n}{2}$, we have the important relations

$$P_{2N}\left(h; \frac{n}{2} - N\right) = P_{2N}(h). \quad (2.7)$$

The same relations hold true for odd n and all $N \geq 1$. For details see [GZ03].

Theorem 2.1. *Assume that $1 \leq N \leq \frac{n}{2}$ for even n and $N \geq 1$ for odd n . Then the identities*

$$\lambda N \sum_{j=0}^N \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) + (\lambda - n + 2N) \sum_{j=0}^N j \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) = 0 \quad (2.8)$$

hold true as identities of rational functions.

The identities (2.8) will be called the *master relations*. A calculation shows that the master relations are equivalent to the identities

$$(\lambda - n + 2N) \sum_{j=0}^N (2N + 2j) \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) = -2N(n - 2N) \sum_{j=0}^N \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) \quad (2.9)$$

of rational functions. In terms of the polynomials

$$Q_{2N}^{res}(\lambda) = -2^{2N} N! \left(\lambda + \frac{n}{2} - 2N + 1 \right) \cdots \left(\lambda + \frac{n}{2} - N \right) \sum_{j=0}^N \mathcal{T}_{2j}^*(\lambda + n - 2N)(v_{2N-2j}) \quad (2.10)$$

and¹

$$\mathcal{V}_{2N}(\lambda) = \left(\lambda + \frac{n}{2} - 2N + 1\right) \cdots \left(\lambda + \frac{n}{2} - N\right) \sum_{j=0}^N (2N+2j) \mathcal{T}_{2j}^*(\lambda+n-2N)(v_{2N-2j}) \quad (2.11)$$

the relations (2.9), in turn, can be formulated as the identities

$$2^{2N-2}(N-1)!\lambda \mathcal{V}_{2N}(\lambda) = \left(\frac{n}{2} - N\right) Q_{2N}^{res}(\lambda) \quad (2.12)$$

of polynomials. These alternative versions will be referred to as master relations, too. The polynomials $Q_{2N}^{res}(\lambda)$ were introduced in [J09a]. They are called the Q -curvature polynomials. The identity (2.12) first appeared in Section 6 of [J09d].

We continue with the

Proof of Theorem 2.1. First assume that n is even. We choose N as stated. For $f \in C_0^\infty(M)$ and $\lambda \notin \{\frac{n}{2} - 1, \dots, 0\}$ we set

$$u = r^\lambda f + \mathcal{T}_2(\lambda)(f)r^{\lambda+2} + \cdots + \mathcal{T}_n(\lambda)(f)r^{\lambda+n}.$$

Then

$$\Delta_{g_+} u + \lambda(n-\lambda)u \in O(r^{\lambda+n+1}). \quad (2.13)$$

We choose small numbers ε and δ so that $0 < \varepsilon < \delta$ and consider the asymptotic expansion of the integral

$$\int_{\varepsilon < r < \delta} \Delta_{g_+} u \, \mathrm{dvol}(g_+) \quad (2.14)$$

for $\varepsilon \rightarrow 0$. Assuming that $\lambda + 2N - n \neq 0$, we determine the coefficient of $\varepsilon^{\lambda+2N-n}$ in this expansion in two different ways. On the one hand, we evaluate the integral

$$\int_{\varepsilon < r < \delta} u \, \mathrm{dvol}(g_+) = \int_\varepsilon^\delta \int_M uv(r)r^{-n-1} dr \, \mathrm{dvol}(h)$$

by plugging in the asymptotic expansions of u and $v(r)$. This yields the contribution

$$-\frac{\varepsilon^{\lambda+2N-n}}{\lambda+2N-n} \left(\sum_{j=0}^N \int_M \mathcal{T}_{2j}(\lambda)(f)v_{2N-2j} \, \mathrm{dvol}(h) \right).$$

On the other hand, Green's formula shows that

$$-\int_{\varepsilon < r < \delta} \Delta_{g_+} u \, \mathrm{dvol}(g_+) = \left(\int_{r=\varepsilon} + \int_{r=\delta} \right) \frac{\partial u}{\partial \nu} r^{-n} v(r) \, \mathrm{dvol}(h),$$

where ν denotes the inward unit normal (with respect to g_+). The coefficient of $\varepsilon^{\lambda+2N-n}$ in the asymptotic of this integral is given by

$$\sum_{j=0}^N (\lambda+2j) \int_M \mathcal{T}_{2j}(\lambda)(f)v_{2N-2j} \, \mathrm{dvol}(h).$$

¹In [J09a] and [J09b], the definition of $\mathcal{V}_{2N}(\lambda)$ involves a shift of the argument.

Now (2.13) implies that

$$-\frac{\lambda(n-\lambda)}{\lambda+2N-n} \sum_{j=0}^N \int_M \mathcal{T}_{2j}(\lambda)(f) v_{2N-2j} \, \text{dvol}(h) = \sum_{j=0}^N (\lambda+2j) \int_M \mathcal{T}_{2j}(\lambda)(f) v_{2N-2j} \, \text{dvol}(h)$$

for all $f \in C_0^\infty(M)$. We apply partial integration and find the pointwise identity

$$-\frac{\lambda(n-\lambda)}{\lambda+2N-n} \sum_{j=0}^N \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) = \sum_{j=0}^N (\lambda+2j) \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}).$$

The latter equation is equivalent to (2.12). Finally, let n be odd and $N \geq 1$. For $f \in C_0^\infty(M)$ and $\lambda \notin \frac{n}{2} - \mathbb{N}$ we set

$$u = r^\lambda f + \mathcal{T}_2(\lambda)(f) r^{\lambda+2} + \cdots + \mathcal{T}_{2N}(\lambda)(f) r^{\lambda+2N}.$$

The assertion follows by analogous consideration of the coefficient of $\varepsilon^{\lambda+2N-n}$ in the asymptotic expansion of (2.14). \square

As an illustration, we make explicit the master relations for $N = 1$ and $N = 2$ for Poincaré-Einstein metrics.

Example 2.1. For $N = 1$, (2.8) reads

$$\lambda(v_2 + \mathcal{T}_2^*(\lambda)(1)) + (\lambda - n + 2) \mathcal{T}_2^*(\lambda)(1) = 0. \quad (2.15)$$

It is straightforward to verify (2.15) by using

$$v_2 = -\frac{1}{2} \mathbf{J} \quad (2.16)$$

and

$$\mathcal{T}_2(\lambda) = \frac{1}{2(n-2-2\lambda)} (\Delta - \lambda \mathbf{J}). \quad (2.17)$$

Example 2.2. For $N = 2$, we have the explicit formulas

$$v_4 = \frac{1}{8} (\mathbf{J}^2 - |\mathbf{P}|^2) \quad (2.18)$$

and

$$\begin{aligned} \mathcal{T}_4(\lambda) = \frac{1}{8(n-2-2\lambda)(n-4-2\lambda)} & \left[(\Delta - (\lambda+2)\mathbf{J})(\Delta - \lambda\mathbf{J}) \right. \\ & \left. + \lambda(2\lambda - n + 2)|\mathbf{P}|^2 + 2(2\lambda - n + 2)\delta(\mathbf{P}d) + (2\lambda - n + 2)(d\mathbf{J}, d) \right] \end{aligned} \quad (2.19)$$

(see [J09a] for details). Direct calculations show that

$$\begin{aligned} & 8\mathcal{T}_4^*(\lambda)(1) + 6\mathcal{T}_2^*(\lambda)(v_2) + 4v_4 \\ &= \left(\frac{n}{2} - 2\right) \frac{1}{(n-2-2\lambda)(n-4-2\lambda)} \left[\lambda(2|\mathbf{P}|^2 - \mathbf{J}^2) + (n-2)(\mathbf{J}^2 - |\mathbf{P}|^2) - \Delta \mathbf{J} \right] \end{aligned}$$

and

$$\begin{aligned} & \mathcal{T}_4^*(\lambda)(1) + \mathcal{T}_2^*(\lambda)(v_2) + v_4 \\ &= -\frac{1}{8}(\lambda - n + 4) \frac{1}{(n-2-2\lambda)(n-4-2\lambda)} [\lambda(2|\mathbf{P}|^2 - \mathbf{J}^2) + (n-2)(\mathbf{J}^2 - |\mathbf{P}|^2) - \Delta \mathbf{J}]. \end{aligned}$$

These results confirm (2.9).

Remark 2.1. *The proof of Theorem 2.1 does not utilize the property that g_+ is Einstein. In fact, the master relations remain true for metrics g_+ of the form*

$$g_+ = r^{-2}(dr^2 + h_r)$$

with even one-parameter families h_r . In particular, for $h_r = h + r^2 h_{(2)} + r^4 h_{(4)} + \dots$, the results of Example 2.1 and Example 2.2 extend if v_2 and v_4 are replaced by

$$v_2 = -\frac{1}{2} \operatorname{tr}(h_{(2)}) \quad \text{and} \quad v_4 = \frac{1}{2} \operatorname{tr}(h_{(4)}) - \frac{1}{4} \operatorname{tr}(h_{(2)}^2) + \frac{1}{8} \operatorname{tr}(h_{(2)})^2,$$

and \mathbf{J} and $|\mathbf{P}|^2$ are replaced by $\operatorname{tr}(h_{(2)})$ and $-4 \operatorname{tr}(h_{(4)}) + 2 \operatorname{tr}(h_{(2)}^2)$.

Some special cases of the master relations are of particular interest. First of all, for $\lambda = 0$, the master relation (2.12) states that

$$Q_{2N}^{res}(0) = 0. \quad (2.20)$$

An independent proof of this result can be found in [BJ09] (Theorem 1.6.6).

The case $\lambda = \frac{n}{2} - N$ will be considered in Section 3. It leads to holographic formulas for Q -curvature.

In the critical case $2N = n$, Theorem 2.1 implies the following vanishing result. We recall that

$$\mathcal{V}_n(\lambda) = \left[\left(\lambda - \frac{n}{2} + 1 \right) \cdots \lambda \right] \sum_{j=0}^{\frac{n}{2}} (n+2j) \mathcal{T}_{2j}^*(\lambda)(v_{n-2j}) \quad (2.21)$$

by (2.11).

Theorem 2.2. $\mathcal{V}_n(\lambda) = 0$.

Theorem 2.2 confirms Conjecture 6.11.2 in [J09a] and shows that the assumption in Theorem 6.11.15 is vacuous.

By the definitions, both polynomials $Q_{2N}^{res}(\lambda)$ and $\mathcal{V}_{2N}(\lambda)$ have degree $\leq N$. However, the master relation (2.12) implies that \mathcal{V}_{2N} , in fact, has only degree $\leq N-1$. The latter fact was proved in [J09d], where it plays a central role in the proof of a universal recursive formula for the Q -curvature Q_8 .

Since (2.8) takes the form

$$2N(2\lambda - n + 2N) \mathcal{T}_{2N}^*(\lambda)(v_0) + \cdots + 2\lambda N v_{2N} = 0,$$

it implies a formula for $P_{2N}^*(\lambda)(1)$ as a linear combination (with coefficients depending on λ) of

$$P_{2j}^*(\lambda)(v_{2N-2j}) \quad \text{for } j = 0, \dots, N-1.$$

In the critical case, this observation was noticed and used in [J09a] for low orders.

3. HOLOGRAPHIC FORMULAS FOR Q -CURVATURES

In the present section, we prove Theorem 1.1. We restate the result in the following form.

Theorem 3.1. *Assume that $1 \leq N \leq \frac{n}{2}$ for even n and $N \geq 1$ for odd n . Then*

$$4Nc_N Q_{2N} = \sum_{j=0}^{N-1} (2N-2j) \mathcal{T}_{2j}^* \left(\frac{n}{2} - N \right) (v_{2N-2j}) \quad (3.1)$$

with

$$c_N = (-1)^N \frac{1}{2^{2N} N! (N-1)!}.$$

Proof. We write (2.8) in the form

$$\sum_{j=0}^{N-1} (2\lambda N + 2(\lambda - n + 2N)j) \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) + 2N(2\lambda - n + 2N) \mathcal{T}_{2N}^*(\lambda)(1) = 0. \quad (3.2)$$

The families $\mathcal{T}_{2j}(\lambda)$ for $j = 0, \dots, N-1$ are regular at $\lambda = \frac{n}{2} - N$. Moreover, the relation

$$\left(\lambda - \frac{n}{2} + N \right) \mathcal{T}_{2N}^*(\lambda) = - \frac{1}{2^{2N} N!} \frac{1}{\left(\frac{n}{2} - \lambda - 1 \right) \cdots \left(\frac{n}{2} - \lambda - N + 1 \right)} P_{2N}^*(\lambda)$$

shows that the product $\left(\lambda - \frac{n}{2} + N \right) \mathcal{T}_{2N}^*(\lambda)$ is regular at $\lambda = \frac{n}{2} - N$. By (2.7) and the self-adjointness of P_{2N} , we have

$$P_{2N}^* \left(\frac{n}{2} - N \right) (1) = P_{2N}^*(1) = P_{2N}(1) = (-1)^N \left(\frac{n}{2} - N \right) Q_{2N}.$$

It follows that the value of $\left(\lambda - \frac{n}{2} + N \right) \mathcal{T}_{2N}^*(\lambda)(1)$ at $\lambda = \frac{n}{2} - N$ equals

$$- \frac{1}{2^{2N} N! (N-1)!} P_{2N}^* \left(\frac{n}{2} - N \right) (1) = (-1)^{N-1} \left(\frac{n}{2} - N \right) \frac{1}{2^{2N} N! (N-1)!} Q_{2N}.$$

Hence for $\lambda = \frac{n}{2} - N$, the master relation (3.2) states that

$$\left(\frac{n}{2} - N \right) \sum_{j=0}^{N-1} (2N-2j) \mathcal{T}_{2j}^* \left(\frac{n}{2} - N \right) (v_{2N-2j}) = (-1)^N \left(\frac{n}{2} - N \right) \frac{4N}{2^{2N} N! (N-1)!} Q_{2N}.$$

Now assume that n is even and $2N < n$. We divide the latter relation by $\frac{n}{2} - N$ and find

$$\sum_{j=0}^{N-1} (2N-2j) \mathcal{T}_{2j}^* \left(\frac{n}{2} - N \right) (v_{2N-2j}) = 4N c_N Q_{2N} \quad (3.3)$$

with

$$c_N = (-1)^N \frac{1}{2^{2N} N! (N-1)!}.$$

This proves the assertion in the subcritical case. The same argument completes the proof in odd dimensions. A formal continuation of (3.3) to $2N = n$ yields the holographic formula

$$\sum_{j=0}^{\frac{n}{2}-1} (n-2j) \mathcal{T}_{2j}^*(0) (v_{n-2j}) = 2n c_{\frac{n}{2}} Q_n \quad (3.4)$$

for the critical Q -curvature Q_n [GJ07], [J09a]. The above proof, however does *not* work in this case since it involves a division by $\frac{n}{2} - N$. In fact, in the critical case, the master relation (2.8) states that

$$\lambda \sum_{j=0}^{\frac{n}{2}} (n+2j) \mathcal{T}_{2j}^*(\lambda) (v_{n-2j}) = 0. \quad (3.5)$$

We have seen above that the left-hand side of the critical master relation (3.5) is regular at $\lambda = 0$ and vanishes at $\lambda = 0$ by trivial reasons. In order to derive (3.4), we differentiate (3.5) at $\lambda = 0$. Separating the last term, we find

$$\sum_{j=0}^{\frac{n}{2}-1} (n+2j) \mathcal{T}_{2j}^*(0) (v_{n-2j}) + 2n (d/d\lambda)|_0 (\lambda \mathcal{T}_n^*(\lambda) (1)) = 0.$$

Now

$$\lambda \mathcal{T}_n(\lambda) = -\frac{1}{2^n \left(\frac{n}{2}\right)!} \frac{1}{\left(\frac{n}{2} - \lambda - 1\right) \cdots (-\lambda + 1)} P_n(\lambda).$$

Hence

$$\sum_{j=0}^{\frac{n}{2}-1} (n+2j) \mathcal{T}_{2j}^*(0) (v_{n-2j}) - \frac{2n}{2^n \left(\frac{n}{2}\right)! \left(\frac{n}{2} - 1\right)!} \dot{P}_n^*(0) (1) = 0.$$

We combine this result with the identity

$$n \left(\dot{P}_n^*(0) - \dot{P}_n(0) \right) (1) = 2^n \left(\frac{n}{2}\right)! \left(\frac{n}{2} - 1\right)! \sum_{j=0}^{\frac{n}{2}-1} 2j \mathcal{T}_{2j}^*(0) (v_{n-2j})$$

(see [GJ07], Proposition 2 or [J09a], Theorem 6.6.4). It follows that

$$\sum_{j=0}^{\frac{n}{2}-1} (n-2j) \mathcal{T}_{2j}^*(0)(v_{n-2j}) = \frac{2n}{2^n \left(\frac{n}{2}\right)! \left(\frac{n}{2}-1\right)!} \dot{P}_n(0)(1).$$

Now the relation (see [GZ03])

$$\dot{P}_n^*(0) = (-1)^{\frac{n}{2}} Q_n \quad (3.6)$$

implies the holographic formula (3.4). \square

We add some comments.

Eq. (3.1) expresses the difference

$$Q_{2N} - (-1)^N 2^{2N-1} N! (N-1)! v_{2N}$$

in terms of the lower order constructions $\mathcal{T}_{2j}(n/2 - N)$ and v_{2j} for $j = 0, \dots, N-1$. The same difference can be expressed also in terms of lower order GJMS-operators and lower order Q -curvatures. For the details we refer to [J09b] and [J09d].

The vanishing result $\mathcal{V}_n(\lambda) = 0$ (see Theorem 2.2) implies

$$n \sum_{j=0}^{\frac{n}{2}} \dot{\mathcal{T}}_{2j}^*(0)(v_{n-2j}) + \sum_{j=0}^{\frac{n}{2}} 2j \dot{\mathcal{T}}_{2j}^*(0)(v_{n-2j}) = 0.$$

In combination with the identity $\dot{Q}_n^{res}(0) = -(-1)^{\frac{n}{2}} Q_n$ (see [GJ07] and [J09a]) we find

$$c_{\frac{n}{2}}^{-1} \frac{1}{n} \sum_{j=0}^{\frac{n}{2}} 2j \dot{\mathcal{T}}_{2j}^*(0)(v_{n-2j}) = -\frac{1}{2} \ddot{Q}_n^{res}(0) - \left(\sum_{j=1}^{\frac{n}{2}-1} \frac{1}{k} \right) Q_n.$$

For an application of the latter relation we refer to Theorem 6.11.15 of [J09a] (see also the remarks around (2.6) in [GJ07]).

We close this section with a brief discussion of two examples of Theorem 3.1.

Example 3.1. *In dimension $n \geq 4$, the holographic formula for Q_4 states that*

$$\frac{1}{4} Q_4 = 4v_4 + 2\mathcal{T}_2^* \left(\frac{n}{2} - 2 \right) (v_2). \quad (3.7)$$

An easy calculation using (2.16)–(2.18) shows that (3.7) is equivalent to the familiar expression

$$Q_4 = \frac{n}{2} J^2 - 2|P|^2 - \Delta J.$$

Example 3.2. *In dimension $n \geq 6$, the holographic formula for Q_6 states that*

$$-\frac{1}{2^6} Q_6 = 6v_6 + 4\mathcal{T}_2^* \left(\frac{n}{2} - 3 \right) (v_4) + 2\mathcal{T}_4^* \left(\frac{n}{2} - 3 \right) (v_2). \quad (3.8)$$

Explicit formulas for $\mathcal{T}_2(\lambda)$ and $\mathcal{T}_4(\lambda)$ are displayed in Example 2.2. For a detailed comparison of (3.8) with alternative explicit formulas for Q_6 we refer to [J09a] (Theorem 6.10.4).

4. MASTER RELATIONS AND Q -CURVATURE POLYNOMIALS FOR ROUND SPHERES

In the present section, we prove the master relations for round spheres \mathbb{S}^n by using hypergeometric identities. More precisely, we establish (2.8) by comparing explicit formulas for both sides of (2.9). These results confirm closed formulas for the Q -curvature polynomials found in [J09b] by different methods.

The arguments rest on the following result.

Proposition 4.1. *On the round sphere \mathbb{S}^n ,*

$$P_{2N}(\lambda)(1) = P_{2N}^*(\lambda)(1) = (-1)^N \left(\frac{n}{2}\right)_N (\lambda)_N \quad (4.1)$$

for all $N \geq 0$. Here $(x)_N = x(x+1) \cdots (x+N-1)$ is the usual Pochhammer symbol.

Proof. On the round sphere \mathbb{S}^n , the operators $P_{2N}(\lambda)$ are polynomials in the Laplacian. In particular, they are self-adjoint. Now, by the definitions, the assertion is equivalent to

$$\mathcal{T}_{2N}(\lambda)(1) = \frac{\left(\frac{n}{2}\right)_N}{2^{2N} N!} \frac{(\lambda)_N}{\left(\lambda - \frac{n}{2} + 1\right)_N} \quad (4.2)$$

for all $N \geq 0$. In fact, it will be more convenient to prove the equivalent assertion that

$$\mathcal{T}_{2N} \left(\frac{1}{4} g_c, \lambda \right) (1) = \frac{\left(\frac{n}{2}\right)_N}{N!} \frac{(\lambda)_N}{\left(\lambda - \frac{n}{2} + 1\right)_N} \quad (4.3)$$

for all $N \geq 0$. Note that (4.3) is equivalent to

$$\sum_{N \geq 0} \mathcal{T}_{2N} \left(\frac{1}{4} g_c; \lambda \right) (1) s^{\lambda+2N} = s^\lambda {}_2F_1 \left(\frac{n}{2}, \lambda; \lambda - \frac{n}{2} + 1; s^2 \right), \quad (4.4)$$

where

$${}_2F_1(a, b; c; x) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

is the Gauß hypergeometric function. In the following, we shall derive (4.4) from the well-known fact that the radial eigenfunctions of the Laplacian of the hyperbolic metric

$$\frac{4}{(1 - |x|^2)^2} \sum_{i=1}^n dx_i^2$$

on the unit ball \mathbb{B}^n with boundary S^{n-1} are constant multiples of

$$u(r) = (1 - r^2)^\lambda {}_2F_1 \left(\lambda, \lambda - \frac{n}{2} + 1; \frac{n}{2}; r^2 \right)$$

(see [BJ09], Section 1.4). The substitution

$$s = \frac{1-r}{1+r}, \quad r = |x|$$

brings the hyperbolic metric into the form

$$s^{-2} \left(ds^2 + \frac{1}{4}(1-s^2)^2 g_c \right).$$

Let

$$v(s) = u \left(\frac{1-s}{1+s} \right).$$

We use the identity (see eq. (1) in Section 2.10 of [E53])

$$\begin{aligned} {}_2F_1(a, b; c; x) &= \alpha {}_2F_1(a, b; a+b-c+1; 1-x) \\ &\quad + \beta (1-x)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-x) \end{aligned}$$

with

$$\alpha = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{and} \quad \beta = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

to write u in the form

$$\begin{aligned} u(r) &= A(1-r^2)^\lambda {}_2F_1 \left(\lambda, \lambda - \frac{n}{2} + 1; 2\lambda - n + 2; 1-r^2 \right) \\ &\quad + B(1-r^2)^{n-1-\lambda} {}_2F_1 \left(\frac{n}{2} - \lambda, n - \lambda - 1; n - 2\lambda; 1-r^2 \right) \end{aligned}$$

with

$$A = \frac{\Gamma(\frac{n}{2})\Gamma(n-2\lambda-1)}{\Gamma(\frac{n}{2}-\lambda)\Gamma(n-\lambda-1)} \quad \text{and} \quad B = \frac{\Gamma(\frac{n}{2})\Gamma(2\lambda-n+1)}{\Gamma(\lambda)\Gamma(\lambda-\frac{n}{2}+1)}.$$

Now we substitute

$$r = \frac{1-s}{1+s}.$$

The relation (see (3.1.11) in [AAR])

$${}_2F_1 \left(a, b; 2b; \frac{4x}{(1+x)^2} \right) = (1+x)^{2a} {}_2F_1 \left(a, a + \frac{1}{2} - b; b + \frac{1}{2}; x^2 \right)$$

implies that

$${}_2F_1 \left(\lambda, \lambda - \frac{n}{2} + 1; 2\lambda - n + 2; \frac{4s}{(1+s)^2} \right) = (1+s)^{2\lambda} {}_2F_1 \left(\lambda, \frac{n-1}{2}; \lambda - \frac{n-3}{2}; s^2 \right)$$

and

$$\begin{aligned} {}_2F_1 \left(\frac{n}{2} - \lambda, n - \lambda - 1; n - 2\lambda; \frac{4s}{(1+s)^2} \right) \\ = (1+s)^{2n-2-2\lambda} {}_2F_1 \left(\frac{n-1}{2}, n - \lambda - 1; \frac{n+1}{2} - \lambda; s^2 \right). \end{aligned}$$

It follows that

$$v(s) = A(4s)^\lambda {}_2F_1\left(\lambda, \frac{n-1}{2}; \lambda - \frac{n-3}{2}; s^2\right) \\ + B(4s)^{n-1-\lambda} {}_2F_1\left(\frac{n-1}{2}, n-\lambda-1; \frac{n+1}{2} - \lambda; s^2\right).$$

But the first term in this sum is a multiple of the right-hand side of (4.4) (after shifting n by one). This completes the proof. \square

Next, we calculate the sum on the right-hand side of (2.9).

Proposition 4.2. *On the round sphere \mathbb{S}^n ,*

$$\sum_{j=0}^N \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) = (-1)^N \frac{\left(\frac{n}{2}\right)_N}{2^{2N} N!} \frac{(\lambda-n+2N)(\lambda-n+1)_{N-1}}{\left(\lambda-\frac{n}{2}+1\right)_N}. \quad (4.5)$$

Proof. The formula

$$g_+ = r^{-2} \left(dr^2 + (1 - r^2/4)^2 g_{S^n} \right)$$

implies $v(r) = (1 - r^2/4)^n$. Thus, the Taylor coefficients of $v(r)$ are given by

$$v_{2N} = (-1)^N \frac{1}{2^{2N}} \binom{n}{N}. \quad (4.6)$$

But since the holographic coefficients are constant, the left-hand side of (4.5) equals

$$\sum_{j=0}^N \mathcal{T}_{2j}^*(\lambda)(1)v_{2N-2j}. \quad (4.7)$$

Thus, by Proposition 4.1 it suffices to verify that

$$\sum_{j=0}^N \frac{\left(\frac{n}{2}\right)_j (\lambda)_j}{2^{2j} j! (\lambda - \frac{n}{2} + 1)_j} (-1)^{N-j} \frac{1}{2^{2N-2j}} \binom{n}{N-j} = (-1)^N \frac{\left(\frac{n}{2}\right)_N}{2^{2N} N!} \frac{(\lambda-n+2N)(\lambda-n+1)}{\left(\lambda-\frac{n}{2}+1\right)_N},$$

i.e.,

$$\sum_{j=0}^N \binom{n}{N-j} (-1)^j \frac{\left(\frac{n}{2}\right)_j (\lambda)_j}{\left(\lambda - \frac{n}{2} + 1\right)_j j!} \frac{1}{j!} = \frac{\left(\frac{n}{2}\right)_N}{N!} \frac{(\lambda-n+2N)(\lambda-n+1)_{N-1}}{\left(\lambda - \frac{n}{2} + 1\right)_N}. \quad (4.8)$$

For the proof of the summation formula (4.8) we write the left-hand side in hypergeometric notation:

$$\binom{n}{N} \sum_{j=0}^N \frac{\left(\frac{n}{2}\right)_N (\lambda)_j}{\left(\lambda - \frac{n}{2} + j\right)_j} \frac{(-N)_j}{(n-N+1)_j j!} \frac{1}{j!} = \binom{n}{N} {}_3F_2\left(\frac{n}{2}, \lambda, -N; \lambda - \frac{n}{2} + 1, n - N + 1; 1\right).$$

In view of

$$\frac{n}{2} + \lambda - N + 2 = \left(\lambda - \frac{n}{2} + 1\right) + (n - N + 1),$$

this is a 2-balanced hypergeometric sum. We evaluate this sum by using Sheppard's formula (see Corollary 3.3.4 in [AAR])

$$\begin{aligned} {}_3F_2(-n, a, b; d, e; 1) &= \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n} \\ &\times {}_3F_2(-n, a, a+b-n-d-e+1; a-n-d+1, a-n-e+1; 1). \end{aligned} \quad (4.9)$$

We find

$$\begin{aligned} &\binom{n}{N} {}_3F_2\left(-N, \frac{n}{2}, \lambda; \lambda - \frac{n}{2} + 1, n - N + 1; 1\right) \\ &= \binom{n}{N} \frac{(\lambda - n + 1)_N (\frac{n}{2} - N + 1)_N}{(\lambda - \frac{n}{2} + 1)_N (n - N + 1)_N} {}_3F_2\left(-N, \frac{n}{2}, -1; n - N - \lambda, -\frac{n}{2}; 1\right) \\ &= \frac{(\frac{n}{2})_N (\lambda - n + 2N)(\lambda - n + 1)_{N-1}}{N! (\lambda - \frac{n}{2} + 1)_N}. \end{aligned}$$

by using

$${}_3F_2\left(-N, \frac{n}{2}, -1; n - N - \lambda, -\frac{n}{2}; 1\right) = 1 + \frac{N}{\lambda - n + N} = \frac{\lambda - n + 2N}{\lambda - n + N}.$$

The proof is complete. \square

Proposition 4.2 implies an explicit formula for the Q -curvature polynomial (2.10).

Corollary 4.1. *On the round sphere \mathbb{S}^n ,*

$$Q_{2N}^{res}(\lambda) = (-1)^{N-1} \prod_{j=0}^{N-1} \left(\frac{n}{2} - j\right) \lambda \prod_{j=1}^{N-1} (\lambda - N - j).$$

Proof. Proposition 4.2 yields

$$Q_{2N}^{res}(\lambda - n + 2N) = (-1)^{N-1} \left(\frac{n}{2}\right)_N (\lambda - n + 2N)(\lambda - n + 1)_{N-1}.$$

This formula implies the assertion. \square

Corollary 4.1 was derived in [J09b] (see Lemma 9.2) by using the factorization identities for Q -curvature polynomials.

Next, we calculate the sum on the left-hand side of (2.9). By (4.2), we obtain

$$\sum_{j=0}^N j \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) = \frac{(-1)^N}{2^{2N}} \binom{n}{N} \sum_{j=0}^N \frac{(\frac{n}{2})_N (\lambda)_j}{(\lambda - \frac{n}{2} + j)_j} \frac{(-N)_j}{(n - N + 1)_j} j \frac{1}{j!}.$$

In terms of hypergeometric notation this sum equals

$$\frac{(-1)^N}{2^{2N}} \binom{n}{N} \frac{\frac{n}{2}\lambda(-N)}{(\lambda - \frac{n}{2} + 1)(n - N + 1)} \times {}_3F_2\left(\frac{n}{2} + 1, \lambda + 1, -N + 1; \lambda - \frac{n}{2} + 2, n - N + 2; 1\right).$$

The ${}_3F_2$ is a 1-balanced hypergeometric sum. By the formula of Pfaff-Saalschütz (see Theorem 2.2.6 in [AAR]) or (4.9), we find

$${}_3F_2\left(\frac{n}{2} + 1, \lambda + 1, -N + 1; \lambda - \frac{n}{2} + 2, n - N + 2; 1\right) = \frac{(\lambda - n + 1)_{N-1}(-\frac{n}{2} + 1)_{N-1}}{(\lambda - \frac{n}{2} + 2)_{N-1}(-n)_{N-1}}.$$

Hence

$$\begin{aligned} \sum_{j=0}^N j \mathcal{T}_{2j}^*(\lambda)(v_{2N-2j}) &= \frac{(-1)^{N-1}}{2^{2N}} \binom{n}{N} \frac{\frac{n}{2}N}{n-N+1} \frac{(-\frac{n}{2}+1)_{N-1}}{(-n)_{N-1}} \frac{\lambda(\lambda-n+1)_{N-1}}{(\lambda-\frac{n}{2}+1)_N} \\ &= \frac{(-1)^{N-1}}{2^{2N}} \frac{(\frac{n}{2})_N}{(N-1)!} \lambda \frac{(\lambda-n+1)_{N-1}}{(\lambda-\frac{n}{2}+1)_N}. \end{aligned}$$

It follows that the second sum in (2.8) equals

$$(-1)^{N-1} \frac{(\frac{n}{2})_N}{2^{2N}(N-1)!} \lambda(\lambda-n-2N) \frac{(\lambda-n+1)_{N-1}}{(\lambda-\frac{n}{2}+1)_N}.$$

On the other hand, by Proposition 4.2, the first sum in (2.8) equals

$$(-1)^N \frac{(\frac{n}{2})_N}{2^{2N}(N-1)!} \lambda(\lambda-n+2N) \frac{(\lambda-n+1)_{N-1}}{(\lambda-\frac{n}{2}+1)_N}.$$

This completes the proof of (2.8) for S^n .

Finally, we note that the discussion in Section 7.9 of [FG07] yields explicit formulas for the families $P_{2N}(\lambda)$ on \mathbb{S}^n .² These formulas can be used to give an alternative proof of Proposition 4.1.

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